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On the initial-value problem for a wavemaker

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The linearized initial-value problem for the generation of straight-crested waves in a deep, inviscid liquid in response to the prescribed motion of a piston wavemaker of finite depth is solved through integral transforms. The indicial admittance (the surface-wave response to a step-function velocity of the wavemaker) is cast in similarity form and expressed in terms of confluent hypergeometric functions for pure (no surface tension) gravity waves. This gravity-wave result, due essentially to Roberts (1987), provides an outer approximation for $x \gg l$ and $gt^2 \gg l$ ($x =$ horizontal distance from wavemaker and $l =$ capillary length) but implies an infinite wave slope at the contact line ($x = 0$) in consequence of the neglect of surface tension. The corresponding similarity solution for capillary waves (no gravity) with either fixed contact angle or fixed contact line is constructed and is found to be analytic in x for $t > 0$ if the contact angle is fixed or singular like $x^4 \log x$ if the contact line is fixed. An inner approximation for gravity waves with either fixed contact angle or fixed contact line is constructed for $x = O(l)$ and $gt^2 \gg l$. The Laplace transform of the general solution is expressed in terms of confluent hypergeometric functions, which permits a compact discussion of its analytical properties.

1. Introduction

Following Roberts (1987) and Joo, Schultz & Messiter (1990), who give references to earlier work, I consider the linearized initial-value problem for the generation of straight-crested surface waves by a wavemaker at the boundary $x = 0$ of an inviscid liquid ($x > 0, y < 0$). [Peregrine (1972) considers the closely related problem of the initial motion of a vertical plate in the reference frame of the plate, but linearization dictates a fixed reference frame in the present context.] Roberts neglects capillarity, whence his results are singular at the contact line (although he attributes the singular behaviour to the neglect of nonlinearity). Joo *et al.* incorporate capillarity in their formulation and show that it relieves the contact-line singularity, but they do not undertake a systematic exploration of capillary effects. Capillary effects also have been considered by Hocking & Mahdmina (1991) in work that is largely complementary to that reported here.

Capillary effects are comparable with gravitational effects in the wavemaker problem if $x = O(l)$ and dominate them if both $x \ll l$ and $t \ll t_c$, where

$$l \equiv (T/g)^{\frac{1}{2}}, \quad t_c = (l/g)^{\frac{1}{2}} \qquad (1.1a, b)$$

are the capillary length and time constant (2.8 mm and 1.7×10^{-2} s for clean water), and T is the kinematic surface tension (the conventional surface tension divided by the density). Their incorporation requires that the capillary pressure be included in the dynamical free-surface condition and that an appropriate condition be invoked at the contact line. The linear approximation to this contact-line condition must be of the form $\eta_x = \mathbb{K}\eta$, where $\eta(x,t)$ is the free-surface displacement and $\mathbb{K}$ is a linear operator (cf. Hocking 1987; Miles 1990); however, $\mathbb{K}$ is unknown for transient motion, and I consider here only the limiting conditions $\eta_x = 0$ ($\mathbb{K} = 0$) or $\eta = 0$ ($\mathbb{K} = \infty$). (Hocking & Mahdmina (1991) posit the contact-line condition $\eta_t = \lambda\eta_x$, which is equivalent to $K = \lambda^{-1}\partial_t$. This is manifestly the simplest plausible interpolation between the limiting conditions $\eta_x = 0$ and $\eta = 0$, but it does not comprehend the time lag that (in my view) must be expected to exist between η_t and η_x.)

Further insight into the respective roles of gravity and capillarity in the present context follows from the conversion of the dispersion relation $\omega^2 = gk + Tk^3$ between the (radian) frequency ω and the wavenumber k to the dimensionless form

$$(\omega t_c)^2 = kl + (kl)^3, \tag{1.2}$$

and the consideration of its roots in the complex-k plane (in anticipation of the subsequent Fourier transformation). It follows from Descartes' rule of signs that (1.2) has one positive root and a pair of complex-conjugate roots, which admit the approximations

$$kl \to (\omega t_c)^2, \pm \mathrm{i} \quad (\omega t_c \to 0), \tag{1.3}$$

and

$$kl \to (\omega t_c)^{\frac{2}{3}}(1, \mathrm{e}^{\frac{2}{3}\mathrm{i}\pi}, \mathrm{e}^{-\frac{2}{3}\mathrm{i}\pi}) \quad (\omega t_c \to \infty). \tag{1.4}$$

The real roots in (1.3) and (1.4) correspond, respectively, to gravity and capillary waves, the imaginary roots in (1.3) correspond to an exponentially decaying disturbance (cf. (1.10)), and the complex-conjugate roots in (1.4) correspond to a decaying (in x) oscillation.

Following Roberts (1987), I suppose that the wavemaker is a piston of depth d.† The linearized initial-value problem then is described by

$$\phi_{xx} + \phi_{yy} = 0 \quad (x > 0, y < 0), \tag{1.5}$$

$$\phi_x = \begin{matrix} u(t) \\ 0 \end{matrix} \quad (y \gtrless -d), \quad \eta_x = \sigma(t) \quad (x = 0), \tag{1.6a, b}$$

$$\phi_y = \eta_t, \quad \phi_t + g\eta = T\eta_{xx} \quad (y = 0), \tag{1.7a, b}$$

$$\phi_y \to 0 \quad (y \downarrow -\infty), \tag{1.8}$$

and

$$\phi|_{y=0} = 0, \quad \eta = 0 \quad (t = 0), \tag{1.9a, b}$$

where $\phi(x,y,t)$ is the velocity potential, $u(t)$ is the prescribed velocity normal to the wavemaker, $\eta(x,t)$ is the free-surface displacement, and $\sigma(t)$ is the wave slope at the contact line. The boundary conditions (1.6a, b) are projected from the displaced position of the wavemaker onto $x = 0$; (1.7a, b) are projected from the displaced surface $y = \eta_0 + \eta$ onto $y = 0$, where η_0 is the static displacement (the meniscus) and admits the linear approximation

$$\eta_0(x) = \eta_0(0)\,\mathrm{e}^{-x/l}. \tag{1.10}$$

The solution of (1.5)–(1.9), for which I develop an integral representation in §2, may be expressed in the form (Duhamel's superposition theorem)

$$\eta(x,t) = \partial_t \int_0^t u(t-\tau)\,\eta_1(x,\tau)\,\mathrm{d}\tau = u(0)\,\eta_1(x,t) + \int_0^t \dot{u}(\tau)\,\eta_1(x,t-\tau)\,\mathrm{d}\tau, \tag{1.11}$$

† The solution of (1.5)–(1.9) for a depth-dependent wavemaker velocity of the separable form $\hat{u}(y,t) = u(t)f(y)$ may be determined through a spatial (y) convolution with the present solution. I simplify much of the subsequent development by letting $d \gg l$, but this is not an essential restriction.

where $\eta_1(x,t)$, *the indicial admittance*, is determined by (1.5)–(1.9) with $u(t)$ therein replaced by the step function

$$u_1(t) = \begin{matrix} 1 \\ 0 \end{matrix} \quad (t \gtrless 0). \tag{1.12}$$

By definition, u_1 is dimensionless, in consequence of which η_1 is dimensionally a time. Moreover, η_1 may comprise the delta-function $\delta(t)$ and its derivatives in consequence of the acceleration $\dot{u}_1(t) = \delta(t)$ implied by (1.12). These singular components do not contribute to η_1 for $t > 0$, but they may be significant in (1.11).

It is expedient in many problems to work with the Laplace transform

$$\bar{\eta}(x,s) \equiv \mathscr{L}\eta(x,t) = s\bar{u}(s)\,\bar{\eta}_1(x,s), \tag{1.13}$$

which follows from (1.11) through the convolution theorem. For example, the steady-state ($t\uparrow\infty$) response to the sinusoidal motion

$$u(t) = a\omega_0 \sin\omega_0 t, \quad \bar{u}(s) = \frac{a\omega_0^2}{s^2+\omega_0^2}, \tag{1.14a,b}$$

is determined by the poles at $s = \pm\mathrm{i}\omega_0$ in the complex-s plane and is given by

$$\eta(x,t) \sim a\omega_0^2\,\mathrm{Re}\,\{\bar{\eta}_1(x,\mathrm{i}\omega_0)\exp(\mathrm{i}\omega_0 t)\} \quad (\omega_0 t\uparrow\infty). \tag{1.15}$$

I obtain the exact solution for $\bar{\eta}_1$ as a Fourier integral in §2 and in terms of confluent hypergeometric functions in §5.

For pure ($T\equiv 0$) gravity waves, $\eta_1(x,t)$ resembles Lamb's (1932, §§238–240) solution of the Cauchy–Poisson problem and may be cast in similarity form (Roberts 1987) and expressed in terms of Fresnel integrals or confluent hypergeometric functions (which possibility appears to have been overlooked by Roberts (1987) and Joo *et al.* (1990)). This similarity solution, which I develop in §3, provides an outer approximation for $x \gg l$ and $t \gg t_c$; in particular,

$$\eta_1 \sim \frac{2t}{\pi}\left[2-\gamma-\ln\left(\frac{gt^2}{d}\right)-4\pi^{\frac{1}{2}}\left(\frac{x}{gt^2}\right)^{\frac{3}{2}}\cos\left(\frac{gt^2}{4x}+\tfrac{1}{4}\pi\right)\right] \quad (l \ll x \ll gt^2 \ll d), \tag{1.16}$$

where $\gamma = 0.577\ldots$ is Euler's constant. Note that (1.16) implies $\eta_{1x} = O(x^{-\frac{1}{2}})$ as $x\downarrow 0$ but is not valid in this limit; nevertheless, it does provide a valid approximation to η_1 at $x=0$.

Gravity may be neglected in $x \ll l$ and $t \ll t_c$, and dimensional analysis then implies a similarity form of η_1 for capillary waves. I construct this solution in §4 and find that

$$\eta_{1a}(x,t) = \frac{2t}{3\pi}\left[\ln\left(\frac{d^3}{Tt^2}\right)+\gamma+2-\tfrac{1}{2}\Gamma(\tfrac{1}{3})\frac{x^2}{(Tt^2)^{\frac{2}{3}}}\right] \quad \left(0 \leqslant \frac{x^3}{l^3} \ll \frac{t^2}{t_c^2} \ll 1\right), \tag{1.17}$$

or

$$\eta_{1b}(x,t) = \frac{3^{\frac{1}{2}}x}{2\pi\Gamma(\frac{2}{3})}\left(\frac{t}{T}\right)^{\frac{1}{3}}\left[\ln\left(\frac{d^3}{Tt^2}\right)+2\psi(\tfrac{4}{3})+3\gamma\right] \quad \left(0 \leqslant \frac{x^3}{l^3} \ll \frac{t^2}{t_c^2} \ll 1\right), \tag{1.18}$$

where the subscript a/b signifies that the contact angle/line is fixed ($\eta_{1x}/\eta_1 = 0$ at $x=0$), and ψ is the logarithmic derivative of the gamma function.

Neither the gravity-wave nor the capillary-wave similarity solution provides an adequate description of $\eta_1(x,t)$ in $x = O(l)$. In §6 and Appendix B, I consider the domain $x = O(l)$ and $l \ll gt^2 \ll d$ and obtain the inner approximations (cf. (1.17) and (1.18))

$$\eta_{1a} = \frac{2}{\pi}t\left\{2-\gamma-\ln\left(\frac{gt^2}{d}\right)+O\left[\left(\frac{x}{gt^2}\right)^2\right]\right\}, \tag{1.19}$$

and
$$\eta_{1b} = \frac{2}{\pi}t\left[2-\gamma-\ln\left(\frac{gt^2}{d}\right)\right](1-e^{-x/l}). \tag{1.20}$$

I make frequent reference to Abramowitz & Stegun (1964) through the prefix AS, followed by the appropriate equation number therein.

2. Transform solution

We attack the initial-value problem through Fourier-cosine transformation with respect to x and Laplace transformation with respect to t. Introducing

$$F(k) = \int_0^\infty f(x)\cos kx\,\mathrm{d}x \equiv \mathscr{F}f, \quad f(x) = \frac{2}{\pi}\int_0^\infty F(k)\cos kx\,\mathrm{d}k \equiv \mathscr{F}^{-1}F, \tag{2.1a, b}$$

$$\bar{f}(s) = \int_0^\infty e^{-st}f(t)\,\mathrm{d}t \equiv \mathscr{L}f, \quad f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\bar{f}(s)\,\mathrm{d}s \equiv \mathscr{L}^{-1}\bar{f} \quad (c>0), \tag{2.2a, b}$$

and transforming (1.5)–(1.9), we obtain

$$\bar{\Phi}_{yy}-k^2\bar{\Phi} = \begin{matrix}\bar{u}\\ 0\end{matrix} \quad (y \gtrless -d), \tag{2.3}$$

$$\bar{\Phi}_y = s\bar{N}, \quad s\bar{\Phi}+(g+Tk^2)\bar{N} = -T\bar{\sigma} \quad (y=0), \tag{2.4a, b}$$

and
$$\bar{\Phi}_y \to 0 \quad (y\downarrow -\infty), \tag{2.5}$$

where $\bar{\Phi} \equiv \mathscr{L}\mathscr{F}\phi$ and $\bar{N} \equiv \mathscr{L}\mathscr{F}\eta$. The solution of (2.3)–(2.5) is given by

$$\bar{\Phi} = -k^{-2}\bar{u}\begin{pmatrix}1-e^{-kd}\cosh ky\\ e^{ky}\sinh kd\end{pmatrix}+k^{-1}s\bar{N}e^{ky} \quad (y \gtrless -d), \tag{2.6}$$

and
$$\bar{N} = (s^2+\omega^2)^{-1}[s\bar{u}k^{-1}(1-e^{-kd})-T\bar{\sigma}k] \quad (\omega^2 \equiv gk+Tk^3), \tag{2.7}$$

where $\bar{\sigma}$ is to be determined through the imposition of the appropriate contact-line condition.

We define $\eta_1(x,t)$ as the response to the step-function velocity (1.12), for which

$$\bar{u}_1(s) = \frac{1}{s}, \quad \bar{\sigma}_1(s) \equiv \frac{\bar{\sigma}}{s\bar{u}}, \quad \bar{\eta}_1 \equiv \frac{\bar{\eta}}{s\bar{u}}, \quad \bar{N}_1 \equiv \frac{\bar{N}}{s\bar{u}} = \frac{k^{-1}(1-e^{-kd})-Tk\bar{\sigma}_1}{s^2+\omega^2}. \tag{2.8a–d}$$

2.1. *Fixed contact angle* ($\sigma = 0$)

Setting $\bar{\sigma}_1 = 0$ (which we signify by the subscript a) in (2.8d) and inverting, we obtain

$$\bar{\eta}_{1a}(x,s) = \frac{2}{\pi}\int_0^\infty\left(\frac{1-e^{-kd}}{k}\right)\frac{\cos kx\,\mathrm{d}k}{s^2+\omega^2}, \quad \eta_{1a}(x,t) = \frac{2}{\pi}\int_0^\infty\left(\frac{1-e^{-kd}}{k}\right)\cos kx\frac{\sin\omega t}{\omega}\mathrm{d}k. \tag{2.9a, b}$$

We remark that η_{1axx} for $d=\infty$ is the solution of the two-dimensional Cauchy–Poisson problem for capillary–gravity waves (Lamb 1932, §269; Rayleigh 1911) and that the asymptotic behaviour of η_{1a} for $x/l > 1.09t/t_c \gg 1$ ($1.09l/t_c$ is the minimum value of the group velocity) may be determined by the method of stationary phase.

The Fourier integral (2.9b) may be expanded in powers of t for $x>0$ by expanding $\omega^{-1}\sin\omega t$ in powers of t and k and inverting term by term, but the result diverges at $x=0$ and therefore is of limited interest in the present context.

2.2. *Fixed contact line*

The assumption of a fixed contact line, $\eta = 0$ at $x = 0$, yields, through (2.1*b*) and (2.8*d*),

$$\bar{\eta}_1(0, s) = \frac{2}{\pi}\int_0^\infty \left[\frac{k^{-1}(1-\mathrm{e}^{-kd}) - T\bar{\sigma}_1(s)\,k}{s^2+\omega^2}\right]\mathrm{d}k = 0, \tag{2.10}$$

for the determination of

$$\bar{\sigma}_1(s) = \frac{\displaystyle\int_0^\infty (1-\mathrm{e}^{-kd})\,k^{-1}(s^2+\omega^2)^{-1}\,\mathrm{d}k}{T\displaystyle\int_0^\infty (s^2+\omega^2)^{-1}\,k\,\mathrm{d}k} = \frac{-\bar{\eta}_{1a}(0,s)}{T\lim\limits_{d\uparrow\infty}\bar{\eta}_{1axx}(0,s)}. \tag{2.11}$$

The Laplace transform of the corresponding indicial admittance (to which we append the subscript b to signify a fixed contact line) is given by

$$\bar{\eta}_{1b}(x, s) = \frac{2}{\pi}\int_0^\infty \left[\frac{k^{-1}(1-\mathrm{e}^{-kd}) - Tk\bar{\sigma}_1(s)}{s^2+\omega^2}\right]\cos kx\,\mathrm{d}k \tag{2.12a}$$

$$= \{1+T\bar{\sigma}_1(s)\lim_{d\uparrow\infty}\partial_x^2\}\,\bar{\eta}_{1a}(x,s), \tag{2.12b}$$

wherein $\bar{\eta}_{1a}(x,s)$ is given by (2.9*a*). The inverse transform of (2.12*b*) may be expressed as a convolution integral, but it is typically necessary to approximate $\bar{\sigma}_1(s)$ prior to inversion.

3. Gravity-wave similarity solution ($T = 0$)

Setting $\omega = (gk)^{\frac{1}{2}}$ and introducing $\theta = (gk)^{\frac{1}{2}}t$, we transform (2.9*b*) to†

$$\eta_1(x,t) = t\,\mathrm{Re}\left\{\mathscr{G}\left(\frac{x+\mathrm{i}d}{gt^2}\right) - \mathscr{G}\left(\frac{x}{gt^2}\right)\right\} \quad (t>0), \tag{3.1}$$

where
$$\mathscr{G}(X) = \frac{4}{\pi}\int_0^\infty (1-\exp(\mathrm{i}X\theta^2)\frac{\sin\theta}{\theta^2}\,\mathrm{d}\theta, \tag{3.2}$$

$$\mathscr{G}'(X) = 4Z\exp(-\tfrac{1}{2}\mathrm{i}\pi Z^2)\,[C(Z)+\mathrm{i}S(Z)] \tag{3.3a}$$

$$= 4Z^2M(1,\tfrac{3}{2},-\tfrac{1}{2}\mathrm{i}\pi Z^2), \quad Z \equiv (2\pi X)^{-\frac{1}{2}}, \tag{3.3b, c}$$

C and S are Fresnel integrals, M is a confluent hypergeometric function, (3.3*b*) follows from (3.3*a*) through AS 7.3.25, $\mathscr{G}(0) = 0$ by construction, and (see Appendix A)

$$\mathscr{G}(X) \sim (2/\pi)\,[\ln(-\mathrm{i}X) + 2 - \gamma] \quad (X\to\infty), \tag{3.4}$$

where $\gamma = 0.577\ldots$ is Euler's constant. Invoking AS 13.1.2 or AS 13.5.1 and integrating with respect to X, subject to $\mathscr{G}(0) = 0$ and (3.4), respectively, we obtain the expansions (cf. Roberts 1987, Appendix B)

$$\mathscr{G}(X) \sim \frac{4}{\pi}\sum_{m=0}^\infty \frac{(2m)!\,(-\mathrm{i}X)^{m+1}}{(m+1)!} - \frac{4}{\pi^{\frac{1}{2}}}\exp\left(-\frac{1}{4}\frac{\mathrm{i}}{X}\right)\sum_{m=1}^\infty \frac{(2m)!}{m!}(\mathrm{i}X)^{m+\frac{1}{2}}, \tag{3.5}$$

† The boundary condition (1.6*b*) on η_x at $x = 0$ must be relaxed, and $T\bar{\sigma}$ is absent from (2.9*a*), for $T = 0$, but η_{1x} then is singular at $x = 0$, and the subscript a would be inappropriate in (3.1).

and
$$\mathscr{G}(X) = \frac{2}{\pi}\left[\ln X + 2 - \gamma - \tfrac{1}{2}\mathrm{i}\pi - \sum_{m=0}^{\infty} \frac{m!}{(2m+3)!}(\mathrm{i}X)^{-m-1}\right]. \quad (3.6)$$

It follows from (3.1), (3.5) and (3.6) and the restrictions $x \gg l$ and $t \gg t_c$ implicit in the neglect of surface tension that

$$\eta_1 \sim \frac{2t}{\pi}\left[2-\gamma+\ln\left(\frac{d}{gt^2}\right) - 4\pi^{\frac{1}{2}}\left(\frac{x}{gt^2}\right)^{\frac{3}{2}}\cos\left(\frac{gt^2}{4x}+\tfrac{1}{4}\pi\right)\right] \quad (l \ll x \ll gt^2 \ll d), \quad (3.7)$$

and
$$\eta_1 \sim \frac{t}{\pi}\left[\ln\left(1+\frac{d^2}{x^2}\right)+\frac{1}{3}\frac{gdt^2}{d^2+x^2}\right] \quad (l \ll gt^2 \ll d, x), \quad (3.8)$$

in agreement with Roberts (1987).

4. Capillary-wave similarity solution ($g = 0$)

4.1. *Fixed contact angle*

Letting $g = 0(\omega^2 = Tk^3)$ and introducing $\theta = (Tk^3)^{\frac{1}{2}}t$ in (2.9 *b*), we place the result in the form (cf. (3.1))

$$\eta_{1a}(x,t) = t\,\mathrm{Re}\{\mathscr{C}(X+\mathrm{i}D) - \mathscr{C}(X)\} \quad (t>0), \quad X = \frac{x}{(Tt^2)^{\frac{1}{3}}}, \quad D = \frac{d}{(Tt^2)^{\frac{1}{3}}}, \quad (4.1\,a\text{–}c)$$

where
$$\mathscr{C}(Z) = \frac{4}{3\pi}\int_0^\infty [1-\exp(\mathrm{i}Z\theta^{\frac{2}{3}})]\frac{\sin\theta}{\theta^2}\mathrm{d}\theta. \quad (4.2)$$

We obtain the Maclaurin expansion of $\mathscr{C}(Z)$ by deforming the path of integration for the $\exp(\pm\mathrm{i}\theta)$ component of $\sin\theta$ in (4.2) to $(0, \pm\mathrm{i}\infty)$ in a complex-θ plane cut along $(-\infty, 0)$, expanding $\exp(\mathrm{i}Z\theta^{\frac{2}{3}})$ in a power series, and integrating term-by-term. The end result is

$$\mathscr{C}(Z) = \frac{4}{3\pi}\sum_{n=1}^{\infty}\frac{\Gamma(\frac{2}{3}n-1)\cos(\frac{1}{3}n\pi)}{n!}(\mathrm{i}Z)^n. \quad (4.3)$$

Letting $Z \to \infty$ in (4.2) and approximating the integral as in Appendix A, we obtain the asymptote (cf. (3.4))

$$\mathscr{C}(Z) \sim (2/\pi)[\ln(-\mathrm{i}Z) + \tfrac{1}{3}(\gamma+2)] \quad (Z \to \infty). \quad (4.4)$$

To complete the asymptotic expansion of $\mathscr{C}$ for $\mathrm{Im}\,Z > 0$, we introduce $\kappa = \theta^{\frac{2}{3}}$ in (4.2), differentiate with respect to Z, expand $\kappa^{-\frac{3}{2}}\sin\kappa^{\frac{3}{2}}$ in powers of κ^3, and integrate term-by-term to obtain

$$\mathscr{C}'(Z) = -\frac{2\mathrm{i}}{\pi}\int_0^\infty \mathrm{e}^{\mathrm{i}\kappa Z}\frac{\sin\kappa^{\frac{3}{2}}}{\kappa^{\frac{3}{2}}}\mathrm{d}\kappa \sim \frac{2}{\pi}\sum_{m=0}^{\infty}\frac{\mathrm{i}^m(3m)!}{(2m+1)!}Z^{-3m-1}, \quad (4.5)$$

the integration of which, subject to (4.4), yields

$$\mathscr{C}(Z) \sim \frac{2}{\pi}\left[\ln(-\mathrm{i}Z) + \tfrac{1}{3}(\gamma+2) - \sum_{m=1}^{\infty}\frac{\mathrm{i}^m(3m-1)!}{(2m+1)!}Z^{-3m}\right] \quad (\mathrm{Im}\,Z > 0). \quad (4.6)$$

If $Z = X$ is real the asymptotic expansion comprises the additional contribution

$$\mathscr{C}_s(X) = 2\pi^{-\frac{1}{2}}(\tfrac{2}{3})^4 X^{-\frac{9}{4}}\exp\left[\mathrm{i}\left(\tfrac{4}{27}X^3 - \frac{3\pi}{4}\right)\right][1+O(X^{-\frac{3}{2}})] \quad (4.7)$$

from the point of stationary phase at $\theta = (\frac{2}{3}X)^3$.

The physical domain of principal interest is $X = O(1)$ and $D \gg 1$, in which $\mathscr{C}(X+\mathrm{i}D)$ and $\mathscr{C}(X)$ in (4.1) may be approximated by (4.4) and (4.3), respectively, to obtain

$$\eta_{1a}(x,t) = \frac{2t}{3\pi}\left\{3\ln D+\gamma+2-2\sum_{n=1}^{\infty}\frac{\Gamma(\frac{4}{3}n-1)\cos(\frac{1}{3}n\pi)X^{2n}}{(2n)!}\right\} \quad (0 \leqslant X \ll D) \tag{4.8a}$$

$$\approx \frac{2t}{3\pi}\left[\ln\left(\frac{d^3}{Tt^2}\right)+\gamma+2-\tfrac{1}{2}\Gamma(\tfrac{1}{3})\frac{x^2}{(Tt^2)^{\frac{2}{3}}}+\ldots\right] \quad (0 \leqslant x^3 \ll Tt^2 \ll d^3). \tag{4.8b}$$

4.2. *Fixed contact line*

Letting $\omega^2 = Tk^3$ and $d^3s^2/T \to \infty$ in (2.11), we obtain

$$\bar{\sigma}_1(s) \sim \frac{3^{\frac{1}{2}}}{2\pi}(Ts^4)^{-\frac{1}{3}}\left[\ln\left(\frac{d^3s^2}{T}\right)+3\gamma\right] \tag{4.9}$$

and

$$\sigma_1(t) \sim \frac{3^{\frac{3}{2}}}{2\pi\Gamma(\frac{1}{3})}\left(\frac{t}{T}\right)^{\frac{1}{3}}\left[\ln\left(\frac{d^3}{Tt^2}\right)+2\psi(\tfrac{4}{3})+3\gamma\right] \quad \left(\frac{Tt^2}{d^3}\downarrow 0\right), \tag{4.10}$$

where ψ is the logarithmic derivative of the gamma function. Combining (4.9) with the Laplace transform of (4.8a),

$$\bar{\eta}_{1a} = \frac{2}{\pi}\left[\frac{\gamma+\ln(dT^{-\frac{1}{3}}s^{\frac{2}{3}})}{s^2}\right]-\frac{2}{3\sqrt{3}}\left[\frac{s^{-\frac{2}{3}}x^2}{T^{\frac{2}{3}}}+\frac{s^{\frac{2}{3}}x^4}{12T^{\frac{4}{3}}}+\ldots\right], \tag{4.11}$$

in (2.12b), we obtain

$$\bar{\eta}_{1b} = \bar{\sigma}_1(s)\,x-\left[\frac{\gamma+\ln(dT^{-\frac{1}{3}}s^{\frac{2}{3}})}{\pi}+\frac{2}{3\sqrt{3}}\right]\frac{x^2}{(Ts)^{\frac{2}{3}}}+\ldots, \tag{4.12}$$

the inversion of which yields

$$\eta_{1b}(x,t) = \frac{t}{3\pi}\left\{\frac{3^{\frac{3}{2}}X}{2\Gamma(\frac{2}{3})}\left[3\ln D+2\psi(\tfrac{4}{3})+3\gamma\right]\right.$$
$$\left.-\frac{X^2}{\Gamma(\frac{2}{3})}\left[3\ln D+2\psi(\tfrac{2}{3})+3\gamma+\frac{2\pi}{\sqrt{3}}\right]+O(X^4\ln D\ln X)\right\}, \tag{4.13}$$

where X and D are defined by (4.1b, c).

5. The Laplace transform $\bar{\eta}_1$

5.1. *Partial-fraction Fourier inversion*

We now introduce the dimensionless variables

$$\mathscr{x} \equiv \frac{x}{l}, \quad \mathscr{d} \equiv \frac{d}{l}, \quad \mathscr{k} \equiv kl, \quad \mathscr{s} \equiv st_c, \quad \bar{\mathscr{h}} \equiv t_c^{-2}\bar{\eta}, \tag{5.1}$$

where l and t_c are defined by (1.1), and transform (2.9a) to

$$\bar{\mathscr{h}}_{1a}(\mathscr{x},\mathscr{s}) = \frac{2}{\pi}\int_0^\infty\frac{(1-\mathrm{e}^{-\mathscr{k}\mathscr{d}})\cos\mathscr{k}\mathscr{x}\,d\mathscr{k}}{D(\mathscr{k})}, \quad D = \mathscr{k}(\mathscr{s}^2+\mathscr{k}+\mathscr{k}^3). \tag{5.2a, b}$$

The Fourier integral (5.2*a*) may be reduced to exponential integrals and their relatives through the partial-fraction expansion

$$\frac{1}{D(k)} = \sum_n a_n\left(\frac{1}{k} - \frac{1}{k+k_n}\right), \quad a_n = \frac{1}{k_n(1+3k_n^2)}, \tag{5.3a, b}$$

where the summation is over the three roots of (cf. (1.2))

$$k_n(k_n^2+1) = s^2 \quad (n = 1, 2, 3). \tag{5.4}$$

We note the limiting approximations (cf. (1.3) and (1.4))

$$k_{1,2,3} \to s^2, \pm \mathrm{i} \quad (s \to 0), \quad k_{1,2,3} \sim s^{\frac{2}{3}}(1, \mathrm{e}^{\pm 2\mathrm{i}\pi/3}) \quad (s \to \infty). \tag{5.5a, b}$$

Substituting (5.3*a*) into (5.2*a*) and invoking

$$\int_0^\infty \left(\frac{1-\mathrm{e}^{-kd}}{k}\right)\cos kx \,\mathrm{d}k = \tfrac{1}{2}\ln\left(1+\frac{d^2}{x^2}\right) \tag{5.6}$$

and
$$\int_0^\infty \frac{\mathrm{e}^{-kd}\cos kx\,\mathrm{d}k}{k+k_n} = \tfrac{1}{2}\exp\left[k_n(d\pm \mathrm{i}x)\right]E_1[k_n(d\pm\mathrm{i}x)], \tag{5.7}$$

where, here and subsequently, the sum of the two terms with alternative signs is implicit (k_n may be complex, so that this sum is not necessarily real), and E_1 is the exponential integral (AS 5.1.1), we obtain

$$\bar{k}_{1a}(x, s) = \pi^{-1}\sum_n \{G[k_n(d\pm\mathrm{i}x)] - G(\pm\mathrm{i}k_n x)\}, \tag{5.8}$$

where
$$G(z) \equiv \gamma + \ln z + \mathrm{e}^z E_1(z) \quad (|\arg z| < \pi) \tag{5.9}$$

is the function introduced by Roberts (1987).

A similar calculation of the dimensionless counterparts of (2.11) and (2.12) yields

$$\hat{\sigma}_1(s) \equiv g\bar{\sigma}_1(s) = \tfrac{1}{2}\pi\bar{k}_{1a}(0, s)/\bar{L}_2(s), \tag{5.10a}$$

where
$$\bar{L}_2(s) = \int_0^\infty \frac{k\,\mathrm{d}k}{s^2+k+k^3} = \sum_n a_n k_n^2 \ln k_n, \tag{5.10b}$$

and
$$\bar{k}_{1b}(x, s) = \bar{k}_{1a}(x, s) + \pi^{-1}\hat{\sigma}_1(s)\sum_n a_n k_n^2\, G''(\pm\mathrm{i}k_n x). \tag{5.11}$$

5.2. *The function G(z)*

$G(z)$ is analytic in a complex-z plane cut along the negative real axis. Invoking the confluent-hypergeometric-function identity (from AS 13.1.6, 13.6.12, 13.6.30)

$$\mathrm{e}^z E_1(z) = U(1,1,z) = -\mathrm{e}^z \ln z + \sum_{m=0}^{\infty} \psi(m+1)\frac{z^m}{m!}, \tag{5.12}$$

where ψ is the logarithmic derivative of the gamma function, we obtain the representations

$$G(z) = \sum_{m=1}^{\infty} [\psi(m+1) - \ln z]\frac{z^m}{m!} \quad (|\arg z| < \pi) \tag{5.13}$$

and
$$G(z) \sim \gamma + \ln z + \sum_{m=0}^{\infty} (-)^m m!\, z^{-m-1} \quad (z \to \infty, |\arg z| < \tfrac{3}{2}\pi). \tag{5.14}$$

The arguments of G in (5.8) are not confined to $(-\pi, \pi)$, whence we require the analytical continuations

$$G(z\,\mathrm{e}^{2\mathrm{i}m\pi}) = G(z) + 2\mathrm{i}m\pi(1-\mathrm{e}^z) \quad (|\arg z| < \pi, m = \pm 1, \pm 2, \ldots) \tag{5.15a}$$

and $$G(x\,\mathrm{e}^{\pm \mathrm{i}\pi}) = \gamma + \ln x - \mathrm{e}^{-x}\,\mathrm{Ei}\,(x) \pm \mathrm{i}\pi(1-\mathrm{e}^{-x}), \tag{5.15b}$$

where x is positive real and $\mathrm{Ei}\,(x)$ is defined by AS 5.1.2. The corresponding continuations of (5.13), but not (5.14), may be obtained by choosing $\ln z = \ln|z| + \mathrm{i}\arg z$ for all $\arg z$.

5.3. *Harmonic motion*

The free-surface displacement for the harmonic motion (1.14) is given by (1.15), which requires the evaluation of $\bar{\eta}_1$ at $s = \mathrm{i}\omega_0(\mathscr{s} = \mathrm{i}\omega_0 t_c)$. The cubic equation (5.4) then admits a real root $\mathscr{k}_1$ and a pair of complex-conjugate roots, $\mathscr{k}_{2,3}$, for which

$$\mathscr{k}_1 = |\mathscr{k}_1|\,\mathrm{e}^{\mathrm{i}\pi} \equiv \mathscr{k}_0\,\mathrm{e}^{\mathrm{i}\pi}, \quad \tfrac{1}{3}\pi < \arg\mathscr{k}_2 = -\arg\mathscr{k}_3 < \tfrac{1}{2}\pi \quad (\mathscr{s} = \omega_0 t_c\,\mathrm{e}^{\frac{1}{2}\mathrm{i}\pi}). \tag{5.16a, b}$$

The corresponding arguments of $\mathscr{k}_{2,3}(d \pm \mathrm{i}x)$ and $\mathscr{k}_1(d - \mathrm{i}x)$ lie in $(-\pi, \pi)$ for all d, $x > 0$, but $\arg\mathscr{k}_1(d+\mathrm{i}x)$ and $\arg\,(\mathrm{i}\mathscr{k}_1 x)$ lie in $(\pi, \frac{3}{2}\pi]$, and (5.15*a*) implies

$$G[\mathscr{k}_1(d+\mathrm{i}x)] = G[\exp\,(-\mathrm{i}\pi)\,\mathscr{k}_0(d+\mathrm{i}x)] + 2\mathrm{i}\pi[1-\exp\,(-\mathscr{k}_0(d+\mathrm{i}x))]. \tag{5.17}$$

Substituting (5.16) and (5.17) into (5.8) and invoking (5.3*b*) for a_n, we obtain

$$\bar{\mathscr{h}}_{1a}(x, \mathrm{i}\omega_0 t_c) = \bar{\mathscr{h}}_{1a*}(x, \mathrm{i}\omega_0 t_c) - \frac{2\mathrm{i}[1-\exp\,(-\mathscr{k}_0 d)]}{\mathscr{k}_0(1+3\mathscr{k}_0^2)}\exp\,(-\mathrm{i}\mathscr{k}_0 x), \tag{5.18}$$

where $\bar{\mathscr{h}}_{1a*}$ is the principal value of $\bar{\mathscr{h}}_{1a}$ and is $O(x^{-2})$ as $x \uparrow \infty$, while the last term in (5.18) represents the radiated wave. Restoring dimensions and invoking (1.2) for $\omega = \omega_0(k_0)$ and (1.15), we obtain [cf. Hocking & Mahdmina (1991) with $\lambda = \infty$ therein]

$$\eta_{1a}(x,t) \sim 2a\left(\frac{1+k_0^2 l^2}{1+3k_0^2 l^2}\right)[1-\exp\,(-k_0 d)]\sin\,(\omega_0 t - k_0 x) \quad (k_0 x, \omega_0 t \uparrow \infty). \tag{5.19}$$

It can be shown that (5.18) and (5.19) are consistent with Havelock's (1929) results for a pure gravity wave, for which $k_0 = \omega_0^2/g$ and the terms for $n = 2, 3$ are omitted from $\bar{\mathscr{h}}_{1a*}$.

The counterpart of (5.19) for a fixed contact line is

$$\eta_{1b}(x,t) \sim 2a\left(\frac{1+k_0^2 l^2}{1+3k_0^2 l^2}\right)\mathrm{Re}\left\{[1-\exp\,(-k_0 d) - k_0^2 l^2 \hat{\sigma}_1(\mathrm{i}\omega_0 t_c)]\exp\,[\mathrm{i}(\omega_0 t - k_0 x - \tfrac{1}{2}\pi)]\right\}, \tag{5.20}$$

where $\hat{\sigma}_1$ is given by (5.10). Anticipating (6.6), we obtain

$$\eta_{1b}(x,t) \sim \eta_{1a}(x,t) + \frac{4a}{\pi}\left\{k_0 l[(\gamma + \ln k_0 d)\sin\,(\omega_0 t - k_0 x) + \pi\cos\,(\omega_0 t - k_0 x)] + O(k_0^2 l^2, k_0^3 l^4 d^{-1})\right\}, \tag{5.21}$$

which is a limiting case of Hocking & Mahdmina's (1991) result.

6. The inner domain for gravity waves: $x = O(l)$, $t \gg t_c$

The behaviour of η_1 for $t \gg t_c$ depends on the behaviour of $\bar{\mathscr{h}}_1$ for $|\mathscr{s}| \ll 1$ in $|\arg\mathscr{s}| < \frac{1}{2}\pi$ (Doetsch 1943), in which domain the substitution of (5.5*a*) into (5.3*b*) and (5.8) yields

$$\pi\bar{\mathscr{h}}_{1a}(x,\mathscr{s}) = \mathscr{s}^{-2}\{G[\mathscr{s}^2(d \pm \mathrm{i}x)] - G(\pm\mathrm{i}\mathscr{s}^2 x)\} + \mathrm{Re}\,\{\mathrm{i}G[\mathrm{i}(d \pm \mathrm{i}x)] - \mathrm{i}G(x\,\mathrm{e}^{\mathrm{i}\pi})\} + O(\mathscr{s}^2), \tag{6.1}$$

where $O(s^m)$, $m \neq 0$, implicitly comprises $O(s^m \ln s)$. The s-dependent terms in (6.1), which represent the contributions of the poles of the Fourier transform at $k = -k_{2,3} = \mp 1$, do not contribute to η_{1a} for $t > 0$, but they are significant for the fixed-contact-line problem (see below).

Substituting (5.13) into (6.1), inverting, and restoring dimensions, we obtain η_{1a}, within an error factor of $1 + O(l^2/g^2t^4)$, in the form (3.1) with $\mathscr{G}$ given by

$$\mathscr{G}(X) \sim \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{(2m)!\,(-\mathrm{i}X)^{m+1}}{(m+1)!}, \tag{6.2}$$

which is identical with the first series in (3.5). The absence of the second series in (3.5) from the present approximation reflects the fact that the approximation (5.5*a*) and the representation (6.1) have not been invoked for large s, where capillarity ensures the exponential vanishing of $\exp(st)\,\bar{\eta}_{1a}(x,s)$ as $s \to \infty$ in $\operatorname{Re} s < 0$ (which would not be so for the approximation (6.1) (cf. Roberts 1987, Appendix B)).

We simplify the further development by assuming $d \gg l$, $x \ll d$, and $l \ll gt^2 \ll d$, in which domain we may approximate $d \pm \mathrm{i}x$ by d, $G(s^2 d)$ by $\gamma + \ln(s^2 d)$, and $G(\mathrm{i}d)$ by $\gamma + \ln d + \frac{1}{2}\mathrm{i}\pi$ and represent $G(\pm \mathrm{i}s^2 x)$ and $G(x\,\mathrm{e}^{\mathrm{i}\pi})$ by (5.13) and (5.15*b*), respectively, to obtain

$$\bar{k}_{1a}(x,s) \sim \frac{2}{\pi} s^{-2}[\gamma + \ln(d s^2)] - x - \mathrm{e}^{-x} + O(s^2 \ln s^2 x, d^{-1}s^4). \tag{6.3}$$

Inverting (6.3) and restoring dimensions, we obtain (cf. (3.7))

$$\eta_{1a} \sim \frac{2t}{\pi}\left[2 - \gamma + \ln\left(\frac{d}{gt^2}\right)\right] \quad (x \ll gt^2, \quad l \ll gt^2 \ll d). \tag{6.4}$$

Turning to the fixed-contact-line problem, we invoke (5.3*b*) and (5.5*a*) in (5.10*b*) to obtain

$$\bar{L}_2(s) = \tfrac{1}{2}\pi + O(s^2 \ln s), \tag{6.5}$$

which may be combined with (6.3) in (5.10*a*) to obtain

$$\hat{\sigma}_1(s) = \frac{2}{\pi} s^{-2}[\gamma + \ln(d s^2)] + O(\ln s). \tag{6.6}$$

Combining the dimensional counterparts of (6.3) and (6.6) in (2.12*b*) and inverting, we obtain (cf. (6.4))

$$\eta_{1b}(x,t) = \frac{2t}{\pi}\left[2 - \gamma + \ln\left(\frac{d}{gt^2}\right)\right](1 - \mathrm{e}^{-x/l}) \quad (x \ll gt^2, \quad l \ll gt^2 \ll d). \tag{6.7}$$

Higher approximations are obtained in Appendix B, where it is shown that η_{1a} is analytic in x for fixed t, but that η_{1b} is singular like $O(x^4 \ln x)$ as $x \to 0$.

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Appendix A. Asymptotic approximation to $\mathscr{G}(X)$

We seek the asymptote of

$$\mathscr{G}(X) = \frac{4}{\pi}\int_0^\infty (1 - \exp(\mathrm{i}X\theta^2))\frac{\sin\theta}{\theta^2}\,\mathrm{d}\theta, \tag{A 1}$$

as $X \to \infty$. Dividing the range of integration into $(0, \theta_1)$ and (θ_1, ∞), where $X^{-\frac{1}{2}} \ll \theta_1 \ll 1$, and invoking

$$I_1 \equiv 2\int_0^{\theta_1} (1-\exp(\mathrm{i}X\theta^2))\frac{\sin\theta}{\theta^2}\,\mathrm{d}\theta \sim 2\int_0^{\theta_1}\left(\frac{1-\exp(\mathrm{i}X\theta^2)}{\theta}\right)\mathrm{d}\theta$$
$$= \mathrm{Ein}(-\mathrm{i}X\theta_1^2) \sim \gamma + \ln(-\mathrm{i}X) + 2\ln\theta_1, \quad \text{(A 2)}$$

where Ein is the modified exponential integral defined in footnote 3 of AS §5.1, and

$$I_2 \equiv 2\int_{\theta_1}^{\infty} (1-\exp(\mathrm{i}X\theta^2))\frac{\sin\theta}{\theta^2}\,\mathrm{d}\theta \sim 2\int_{\theta_1}^{\infty}\frac{\sin\theta}{\theta^2}\,\mathrm{d}\theta$$
$$= 2\left[\frac{\sin\theta_1}{\theta_1} - \mathrm{Ci}(\theta_1)\right] \sim 2(1-\gamma-\ln\theta_1), \quad \text{(A 3)}$$

we obtain
$$\mathscr{G}(X) = \frac{2}{\pi}(I_1+I_2) \sim \frac{2}{\pi}[\ln(-\mathrm{i}X)+2-\gamma]. \quad \text{(A 4)}$$

Appendix B. The expansion about $x = 0$

We now assume $x = O(1)$ and $\mathscr{d} \gg 1$, in which domain $\mathscr{d} \pm \mathrm{i}x$ may be approximated by $\mathscr{d}$ in (5.8) and $G(\mathscr{k}_n \mathscr{d})$ and $G(\pm\mathrm{i}\mathscr{k}_n x)$ represented by (5.14) and (5.13), respectively, to obtain

$$\tfrac{1}{2}\pi\bar{\mathscr{k}}_{1a}(x,s) = \tfrac{1}{2}\pi\bar{\mathscr{k}}_{1a}(0,s) + \sum_{m=1}^{\infty}\frac{(-)^m}{(2m)!}\bar{L}_{2m}x^{2m}$$
$$+\sum_{m=1}^{\infty}\frac{(-)^m}{(2m)!}\bar{K}_{2m}[\ln x - \psi(2m+1)]\,x^{2m} + \tfrac{1}{2}\pi\sum_{m=0}^{\infty}\frac{(-)^{m-1}\bar{K}_{2m+1}}{(2m+1)!}x^{2m+1} \quad (x \ll \mathscr{d}), \quad \text{(B 1)}$$

where

$$\tfrac{1}{2}\pi\bar{\mathscr{k}}_{1a}(0,s) = \bar{K}_0(s)(\gamma+\ln\mathscr{d}) + \bar{L}_0(s) + \sum_{m=1}^{\infty}(-)^{m-1}(m-1)!\,\bar{K}_{-m}(s)\,\mathscr{d}^{-m}, \quad \text{(B 2)}$$

$$\bar{K}_m(s) \equiv \sum_n a_n \mathscr{k}_n^m, \quad \bar{L}_m(s) \equiv \sum_n a_n \mathscr{k}_n^m \ln\mathscr{k}_n. \quad \text{(B 3}a, b\text{)}$$

The corresponding approximation to $\bar{\mathscr{k}}_{1b}(x,s)$ is given by (5.10) and (5.11).

It follows directly from (5.2b), which implies $D \sim \mathscr{k}^4$ as $\mathscr{k} \to \infty$, and (5.3a) that

$$\bar{K}_0 = s^{-2}, \quad \bar{K}_1 = 0, \quad \bar{K}_2 = 0, \quad \bar{K}_3 = 1, \quad \text{(B 4}a\text{–}d\text{)}$$

while (5.4) implies the recursion equation

$$\bar{K}_{m+1} + \bar{K}_{m+3} = s^2\bar{K}_m, \quad (m = \pm1, \pm2, \ldots). \quad \text{(B 5)}$$

The solution of (B 5), subject to (B 4), is given by

$$(-)^{m-1}\bar{K}_{2m} = [m-2]\,s^2 - \tfrac{1}{24}m[m-5][m-4]^2 s^6 + \ldots \quad (m = 1, 2, \ldots), \quad \text{(B 6}a\text{)}$$
$$(-)^{m-1}\bar{K}_{2m+1} = 1 - \tfrac{1}{2}[m-3][m-2]\,s^4 + \ldots \quad (m = 1, 2, \ldots), \quad \text{(B 6}b\text{)}$$

and

$$\bar{K}_{-m} = s^{-2m-2}\{1 + [m-2]\,s^4 + \tfrac{1}{2}[m-5][m-4]\,s^8 + \ldots\} \quad (m = 0, 1, \ldots), \quad \text{(B 6}c\text{)}$$

where
$$[m] = m \quad (m \geqslant 0), \quad [m] = 0 \quad (m \leqslant 0). \quad \text{(B 7}a, b\text{)}$$

It follows from (B 6a, b) that the inverse transforms of the corresponding terms in (B 1) vanish for $t > 0$† and hence that the x-dependent component of $\bar{k}_{1a}$ is derived entirely from the series in $\bar{L}_{2m}x^{2m}$, which is analytic in x.

Substituting (B 6) into (B 2) and invoking $d \gg 1$ (but $s^2 d$ may be $O(1)$, we obtain

$$\tfrac{1}{2}\pi\bar{k}_{1a}(0,s) = s^{-2}\left[\gamma + \ln d + \sum_{m=1}^{\infty}(-)^{m-1}(m-1)!\,(s^2 d)^{-m}\right] + \bar{L}_0(s), \tag{B 8}$$

where, here and subsequently, the terms in $d^{-m}(m \geq 1)$ are multiplied by an implicit error factor of $1 + O(d^{-2})$.

Tractable approximations to $\bar{L}_m(s)$ appear to require expansions about either $s = 0$ or $s = \infty$, starting from either (5.5a) or (5.5b), which yield

$$\bar{L}_{2m} = 2s^{2(2m-1)}[1-2(m+1)\,s^4 + O(s^8)]\ln s + (-)^{m-1}(\tfrac{1}{2}\pi + \tfrac{1}{2}s^2) - s^{2(2m+1)} + O(s^4) \quad (s \to 0) \tag{B 9a}$$

and

$$\bar{L}_{2m} = \tfrac{2}{3}\bar{K}_{2m}\ln s + \frac{4\pi}{9}\sin(\tfrac{2}{3}m\pi)\,s^{2m/3} + O(s^{\frac{4}{3}m-\frac{10}{3}}) \quad (s \to \infty). \tag{B 9b}$$

The latter approximation leads to results that are equivalent to those of §4 through $O(x^4)$ in the limit $t \downarrow 0$ and need not be considered further. Substituting (B 9a) into (B 1) and (B 8), inverting, and restoring dimensions, we obtain

$$\eta_{1a}(0,t) = \frac{2t}{\pi}\left\{2-\gamma+\ln\left(\frac{d}{gt^2}\right) + \sum_{m=1}^{\infty}\frac{(-)^{m-1}(m-1)!}{(2m+1)!}\left(\frac{gt^2}{d}\right)^m + 8\left(\frac{l}{gt^2}\right)^2 + O\left[\left(\frac{l}{gt^2}\right)^4\right]\right\}, \tag{B 10}$$

and

$$\eta_{1a}(x,t) = \eta_{1a}(0,t) + \frac{4t}{\pi}\sum_{m=1}^{\infty}\frac{(-)^{m-1}}{(2m)!}\left(\frac{x}{gt^2}\right)^{2m}\left\{(4m-2)! - 2(m+1)(4m+2)!\left(\frac{l}{gt^2}\right)^2 + O\left[\left(\frac{l}{gt^2}\right)^4\right]\right\}, \tag{B 11}$$

which is analytic in x.

Turning to the fixed-contact-line problem, we combine (B 8) and (B 9a) in (5.10a) to obtain

$$\bar{\sigma}_1(s) = \frac{2}{\pi}s^{-2}\left[1-\frac{4}{\pi}s^2(\ln s + \tfrac{1}{4}) + O(s^4)\right]\left[\gamma + \ln d + 2\ln s - \tfrac{1}{2}\pi s^2 + \sum_{m=1}^{\infty}(-)^{m-1}(m-1)!\,(s^2 d)^{-m}\right], \tag{B 12}$$

the inversion of which, followed by the restoration of dimensions, yields

$$\sigma_1(t) = l^{-1}\eta_{1a}(0,t) - \frac{8}{\pi^2}(gt)^{-1}\left\{3\gamma - \tfrac{1}{2} - \ln\left(\frac{d}{gt^2}\right) + \ln\left(\frac{gt^2}{l}\right) + \sum_{m=1}^{\infty}\frac{(-)^{m-1}(m-1)!}{(2m-1)!}\left(\frac{gt^2}{d}\right)^{2m}\left[\psi(2m) + \tfrac{1}{4} - \tfrac{1}{2}\ln\left(\frac{gt^2}{l}\right)\right] + O\left(\frac{l}{gt^2}\right)\right\}. \tag{B 13}$$

† But non-negative, integral powers of s in $\bar{\eta}_1$ may be significant for the inversion of $s\bar{u}\bar{\eta}_1$ — in particular, if $u = O(t^p)$, $p \geq 1$, as $t \downarrow 0$. See Lighthill (1959) for the inversion of terms like s^n and $s^n \log s$ for $n \geq 0$.

The corresponding approximation to η_{1b} may be calculated through either (2.12*b*) or (5.11). Approximations of higher order than (6.7) are complicated, but it is worth emphasizing that η_{1b} is singular at $O(x^4 \ln x)$ as $x \downarrow 0$ (cf. (4.13)).

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